Optimal Fertility along the Lifecycle

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The postponement of births: empirical evidence (1)

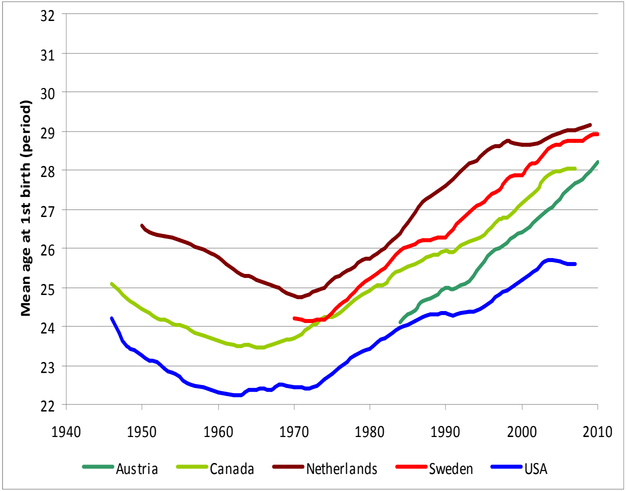


Fig. 1: Average age of women at first birth.

The postponement of births: empirical evidence (2)

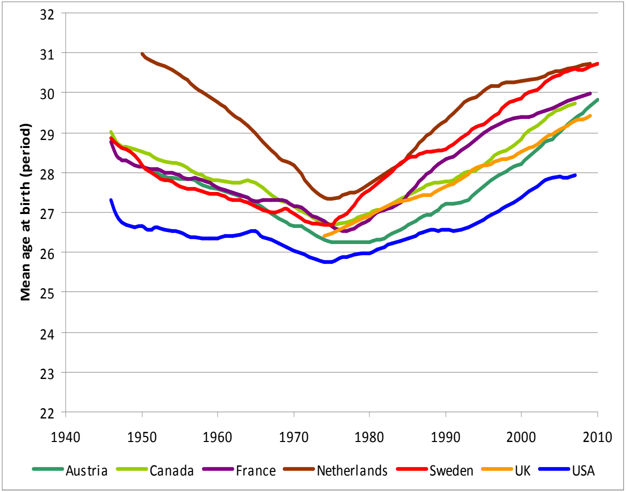


Fig. 2: Average age of women at birth (all births).

The postponement of births: empirical evidence (3)

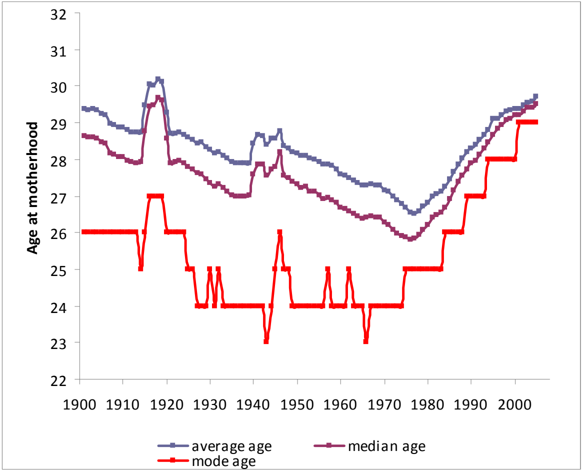


Fig. 3: Average, mode and median age at motherhood, France.

Questions and related research

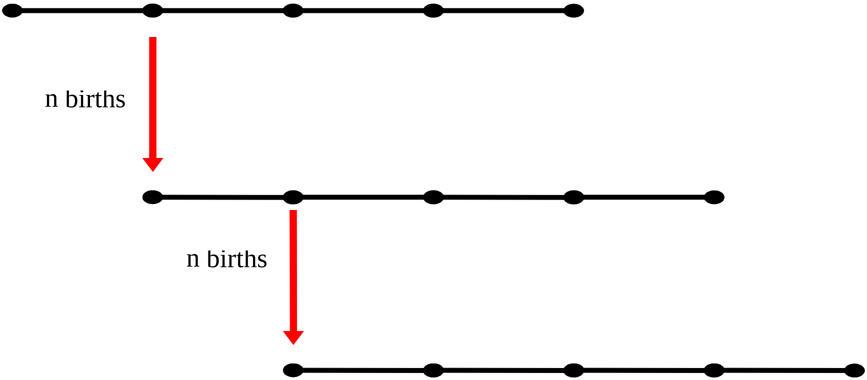
1. What are the **causes** of that postponement of births?
2. What are its **effects** on macroeconomic dynamics?
3. Is the postponement of births **optimal**?

- Question 1 is largely studied in the literature (Gustaffson 2001).
  - Happel *et al* (1984): consumption smoothing over the lifecycle.
  - Cigno & Ermisch (1989): opportunity costs in terms of education.
- In this paper, we focus on questions 2 and 3.
  - D'Albis *et al* (2010) OLG with TFR declining in timing of births:
    - there exists a monetary steady-state if the average age of consumers is larger than the average age of producers.
    - the optimal growth rate of population at the steady-state is larger than the one at the monetary equilibrium.

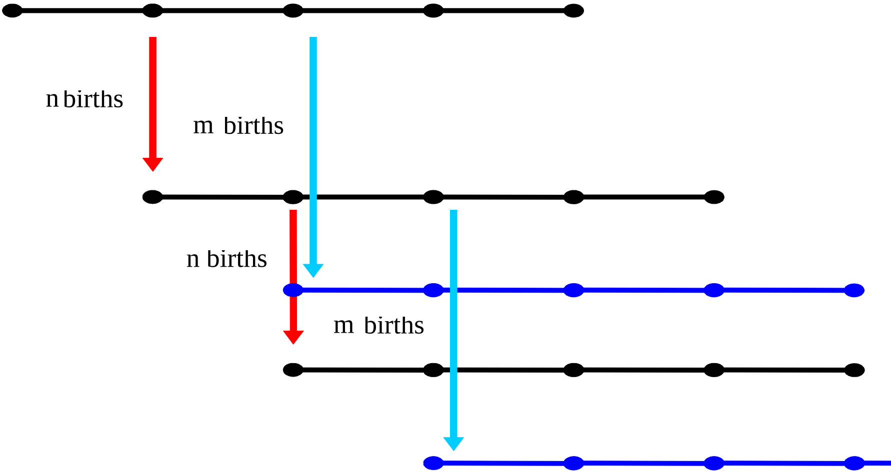
This paper

- We develop a four-period OLG model with physical capital.
- Two reproduction periods (instead of one).
- Early fertility n + late fertility m = total fertility (TFR).
- Individuals take factor prices as *given*.
- Baseline: fertility timing taken as *given* (relaxed later on).
- Our questions:
  1. Is the dynamics varying with the timing of births for a *given* TFR?
  2. What is the *optimal* fertility timing? Effect on Golden Rule?
  3. Does Samuelson's *Serendipity Theorem* still hold?
  4. Are those answers robust to the model we use?

From standard OLG models...



...to a richer demographic structure



Our results

1. The timing of births matters a lot for long-run dynamics, even for a *given* total fertility rate (TFR).

   The major issue is whether the cohort growth factor $g_t \equiv \frac{N_t}{N_{t-1}}$ converges or not towards a constant $g = \frac{n + \sqrt[2]{n^2 + 4m}}{2}$ in the long-run.

2. The long-run social optimum allows for various pairs (n, m), as long as $\frac{n + \sqrt[2]{n^2 + 4m}}{2} = g^*$. No one-to-one substituability between n and m. $=>$ the TFR $n + m$ is irrelevant.

3. An Extended Serendipity Theorem: if a government imposes (n, m) such that $g = g^*$, the competitive economy converges towards the long-run social optimum.

4. Overall robustness of result 1. to behavioural assumptions, fertility choices, number of reproduction periods.

Outline

1. Fertility timing and long-run dynamics
   - Population dynamics
   - Demo-economic dynamics (myopic anticipations)

2. Long-run social optimum
   - Optimal fertility and fertility timing
   - The Serendipity Theorem

3. Extension and robustness checks
   - Rational anticipations
   - Endogenous fertility
   - Three reproduction periods

4. Conclusions

The model

- 4-period OLG model:
  - period 1: childhood;
  - periods 2 and 3: labour, consumption, savings and reproduction;
  - period 4: retirement and consumption.
- Initial conditions: $N_{-1} > 0$, $N_0 > 0$, where N_t denotes the number of individuals born at period t.
- Two reproduction periods: $n \geq 0$ births in period 2 and $m \geq 0$ births in period 3.
- The number of individuals born at time t is:

$$N_t = nN_{t-1} + mN_{t-2}$$

- Hence the growth factor of cohort size g_t is given by:

$$g_t \equiv \frac{N_t}{N_{t-1}} = n + m\frac{N_{t-2}}{N_{t-1}} = n + \frac{m}{g_{t-1}} = f(g_{t-1})$$

A condition for demographic convergence

- Asymptotic convergence of g_t towards $\frac{n+\sqrt[2]{n^2+4m}}{2}$ if and only if $n > 0$.

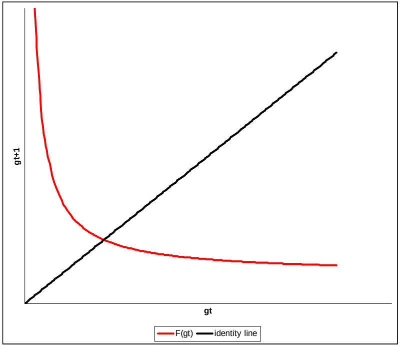


Fig. 4: The long-run g_t

- When $n = 0$, $|f'(g)| = \left|\frac{-m}{g^2}\right| = 1$, violating stability condition.

Demographic dynamics: two polar cases

- When $n > 0$ and $m = 0$, g_t grows or declines at a constant rate:

$$g_1 = g_2 = ... = g_\infty = n$$

- When $n = 0$ and $m > 0$, g_t exhibits a 2-period cycle:

$$\begin{aligned}
g_1 &= \frac{m}{g_0} \\
g_2 &= \frac{m}{g_1} = g_0 \\
g_3 &= \frac{m}{g_2} = \frac{m}{g_0} = g_1 \\
g_4 &= \frac{m}{g_3} = \frac{m}{g_1} = g_0 \\
g_5 &= \frac{m}{g_4} = \frac{m}{g_0} = g_1 \\
&\dots
\end{aligned}$$

- Contrast with Lotka Theorem (1939) in continuous time (see *infra*).

Demographic dynamics: two polar cases

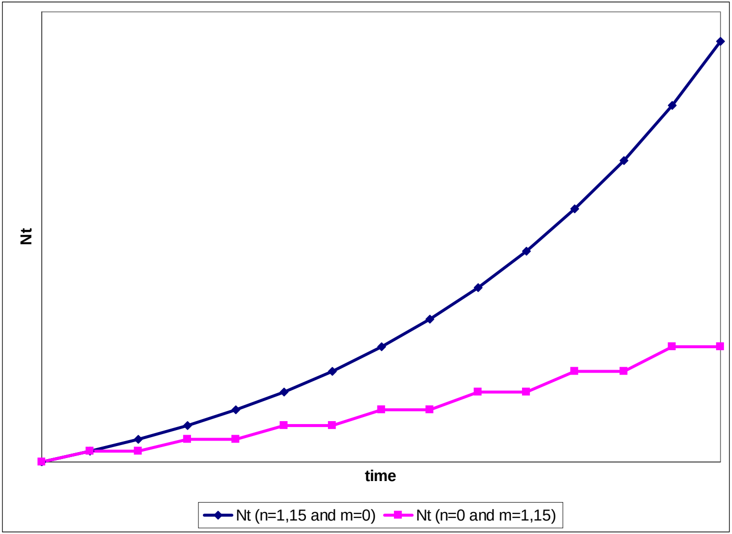


Figure: Number of births under distinct fertility timing.

Demographic dynamics: impact on labour force (1)

- Total labour force at t is:

$$L_t = N_{t-1} + N_{t-2} = g_{t-1} N_{t-2} + N_{t-2}$$

- Dividing it by $L_{t-1} = N_{t-2} + N_{t-3}$ yields the labour growth factor:

$$\frac{L_t}{L_{t-1}} = \frac{g_{t-1} N_{t-2} + N_{t-2}}{N_{t-2} + N_{t-3}} = g_{t-2} \frac{1 + g_{t-1}}{1 + g_{t-2}}$$

- **If $\mathbf{n > 0}$**, g_t converges towards $\frac{n + \sqrt[2]{n^2 + 4m}}{2}$ in the long-run. $\frac{n + \sqrt[2]{n^2 + 4m}}{2}$ is also the long-run labour growth factor.
- **If $\mathbf{n = 0}$**, there is, in general, no convergence. Since $g_{t-2} \times g_{t-1} = m$, the labour force growth ratio is, in that case:

$$\frac{L_t}{L_{t-1}} = \frac{m(1 + g_{t-1})}{m + g_{t-1}}$$

- Labour growth fluctuates, except when $m = 1$ (replacement fertility).

Demographic dynamics: impact on labour force (2)

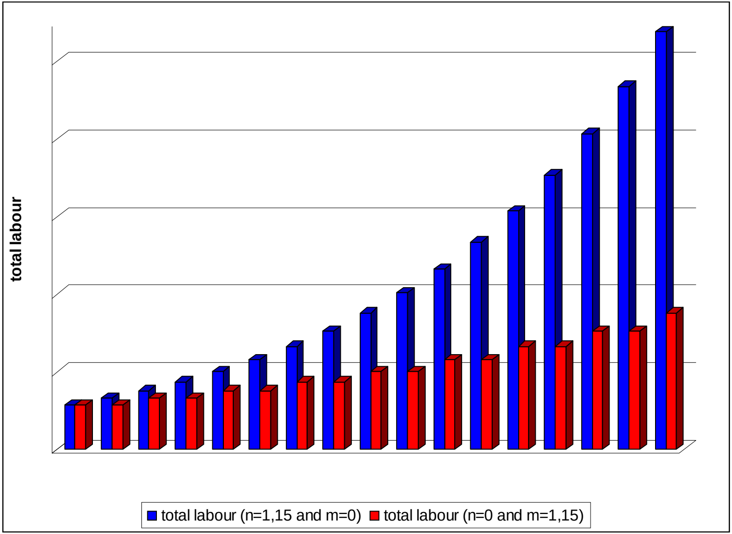


Figure: Total labour under distinct fertility timing.

Production

- The production of an output Y_t involves capital K_t and labour L_t, according to the function:

$$Y_t = F(K_t, L_t) = \bar{F}(K_t, L_t) + (1-\delta)K_t$$

where δ is the depreciation rate of capital, and where $\bar{F}(K_t, L_t)$ is homogeneous of degree one.

- The production process can be rewritten in intensive terms as:

$$y_t = F\left(k_t, 1 + \frac{N_{t-2}}{N_{t-1}}\right)$$

where $y_t = \frac{Y_t}{L_t^y} = \frac{Y_t}{N_{t-1}}$ and $k_t = \frac{K_t}{L_t^y} = \frac{K_t}{N_{t-1}}$.

- Factors are paid at their marginal productivities: wage w_t for labour and savings return R_t for capital.

Savings decision

- Agents solve the problem:

$$\max_{c_t, d_{t+1}, b_{t+2}} u(c_t) + \beta u(d_{t+1}) + \beta^2 u(b_{t+2})$$

$$\text{s.t.} w_t + \frac{w_{t+1}}{R_{t+1}} = c_t + \frac{d_{t+1}}{R_{t+1}} + \frac{b_{t+2}}{R_{t+1} R_{t+2}}$$

- Hence the capital accumulation equation is:

$$k_{t+1} = \frac{s\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right)}{g_t} + \frac{z\left(R_t, R_{t+1}, w_{t-1}, w_t\right)}{g_{t-1} g_t}$$

where $s_t \equiv s\left(\cdot\right)$, $z_{t+1} \equiv z\left(\cdot\right)$ are 2nd- and 3rd-period savings.

- We focus on equilibria under *myopic anticipations*. Hence:

$$k_{t+1} = \frac{\sigma\left(k_t\right)}{g_t} + \frac{\zeta(k_{t-1})}{g_{t-1} g_t}$$

where $s_t = s\left(R(k_t), R(k_t), w(k_t), w(k_t)\right) \equiv \sigma\left(k_t\right)$ and $z_{t+1} = z\left(R(k_t), R(k_t), w(k_t), w(k_t)\right) \equiv \zeta(k_t)$.

Long-run dynamics: the system

- The dynamics of the economy is described by the following three-dimensional first-order dynamic system:

$$\begin{aligned}
k_{t+1} &\equiv G(k_t, \Omega_t, g_t) = \frac{\sigma(k_t)}{g_t} + \frac{\Omega_t}{g_t} \\
\Omega_{t+1} &\equiv H(k_t) = \frac{\zeta(k_t)}{g_t} \\
g_{t+1} &\equiv I(g_t) = n + \frac{m}{g_t}
\end{aligned}$$

Long-run dynamics: general results

Proposition

Assume that $\sigma(0) = 0$, $\sigma'(k_t) > 0$, $\zeta(0) = 0$ *and* $\zeta'(k_t) > 0$. *Assume* $n + m > 0$. *Denote* $\sqrt[2]{n^2 + 4m}$ *by* Ψ.

- *If* $\sigma(0) = 0$, $\zeta(0) = 0$, $lim_{k\to 0} \frac{\Psi+n}{2}\left[1 - \frac{2\sigma'(k_t)}{n+\Psi}\right] < lim_{k\to 0} \frac{2\zeta'(k_t)}{n+\Psi}$ *and* $lim_{k\to +\infty} \frac{\Psi+n}{2}\left[1 - \frac{2\sigma'(k_t)}{n+\Psi}\right] > lim_{k\to +\infty} \frac{2\zeta'(k_t)}{n+\Psi}$, *there exists a stationary equilibrium.*
- *That stationary equilibrium is locally stable if and only if:*

(i) $\frac{16m\zeta'(k)}{(n+\Psi)^4} < 1$

(ii) $1 >$
$$\left[\frac{-4\zeta'(k)}{(n+\Psi)^2} - \frac{8m\sigma'(k)}{(n+\Psi)^3}\right] - \left[\frac{2\sigma'(k)}{n+\Psi} - \frac{4m}{(n+\Psi)^2}\right]\left[\frac{16m\zeta'(k)}{(n+\Psi)^4}\right] + \left[\frac{16m\zeta'(k)}{(n+\Psi)^4}\right]^2$$

(iii) $\frac{4\zeta'(k)}{(n+\Psi)^2} + \frac{8m\sigma'(k)}{(n+\Psi)^3} - 1 < \frac{2\sigma'(k)}{n+\Psi} - \frac{4m}{(n+\Psi)^2} + \frac{16m\zeta'(k)}{(n+\Psi)^4} <$
$\frac{-4\zeta'(k)}{(n+\Psi)^2} - \frac{8m\sigma'(k)}{(n+\Psi)^3} + 1$

Long-run dynamics: two polar cases

Corollary

- *Assume* $n > 0$ *and* $m = 0$. *Provided* $\sigma(0) = 0$, $\zeta(0) = 0$, $lim_{k\to 0}\ n\left[1 - \frac{\sigma'(k_t)}{n}\right] < lim_{k\to 0}\ \frac{\zeta'(k_t)}{n}$ *and* $lim_{k\to +\infty}\ n\left[1 - \frac{\sigma'(k_t)}{n}\right] > lim_{k\to +\infty}\ \frac{\zeta'(k_t)}{n}$, *there exists a stationary equilibrium. Provided* $\frac{\zeta'(k)}{n^2} - 1 < \frac{\sigma'(k)}{n} < -\frac{\zeta'(k)}{n^2} + 1$, *that equilibrium is locally stable.*
- *Assume* $n = 0$ *and* $m > 0$. *Provided* $\sigma(0) = 0$, $\sigma'(k_t) > 0$, *as well as* $\zeta(0) = 0$, $\zeta'(k_t) > 0$, *we have that, if* $lim_{k\to 0}\ \sqrt[2]{m}\left[1 - \frac{\sigma'(k_t)}{\sqrt[2]{m}}\right] < lim_{k\to 0}\ \frac{\zeta'(k_t)}{\sqrt[2]{m}}$ *and* $lim_{k\to +\infty}\ \sqrt[2]{m}\left[1 - \frac{\sigma'(k_t)}{\sqrt[2]{m}}\right] > lim_{k\to +\infty}\ \frac{\zeta'(k_t)}{\sqrt[2]{m}}$, *there exists a stationary equilibrium. That equilibrium, if it exists, is necessarily unstable.*

Long-run dynamics when n = 0

Proposition

Denote $\hat{D}(k_t) \equiv g_0 \left[\sigma^{-1}\left(\frac{m}{g_0}\left(k_t - \frac{\zeta(k_t)}{m}\right)\right) - \frac{\sigma(k_t)}{g_0}\right]$, $\hat{E}(k_t) \equiv \frac{\zeta(\frac{\sigma(k_t)}{g_0} + \frac{\Omega_t}{g_0})}{g_0}$,
$\check{D}(k_t) \equiv \frac{m}{g_0}\left[\sigma^{-1}\left(\left(k_t - \frac{\zeta(k_t)}{m}\right)g_0\right) - \frac{g_0\sigma(k_t)}{m}\right]$, $\check{E}(k_t) \equiv g_0 \frac{\zeta(\frac{g_0\sigma(k_t)}{m} + \frac{g_0\Omega_t}{m})}{m}$.

- *If* $lim_{k\to 0}\hat{E}'(k_t) > lim_{k\to 0}\hat{D}'(k_t)$, $lim_{k\to\infty}\hat{E}'(k_t) < lim_{k\to\infty}\hat{D}'(k_t)$, $lim_{k\to 0}\check{E}'(k_t) > lim'_{k\to 0}\check{D}(k_t)$ *and* $lim_{k\to\infty}\check{E}'(k_t) < lim_{k\to\infty}\check{D}'(k_t)$, *the long-run dynamics is a two-period cycle* $(\hat{k}, \hat{\Omega}, g_0), (\check{k}, \check{\Omega}, \frac{m}{g_0})$.
- *Convergence to the cycle* $(\hat{k}, \hat{\Omega}, g_0), (\check{k}, \check{\Omega}, \frac{m}{g_0})$ *arises, iff:*

$$\left|\frac{\hat{Q}}{2} \pm \sqrt[2]{\frac{\hat{Q}^2 m g_0^2 - 4\zeta'(\frac{\sigma(\hat{k})+\hat{\Omega}}{g_0})\zeta'(\hat{k})}{4mg_0^2}}\right|, \left|\frac{\check{Q}}{2} \pm \sqrt[2]{\frac{\check{Q}^2 m^3 - 4\zeta'(\frac{g_0\sigma(\check{k})+g_0\check{\Omega}}{m})\zeta'(\check{k})}{4m^3}}\right| < 1,$$

with $\hat{Q} \equiv \left[g_0^2\left(\sigma'\left(\frac{\sigma(\hat{k})+\hat{\Omega}}{g_0}\right)\sigma'(\hat{k}) + \zeta'(\hat{k})\right) + m\zeta'(\frac{\sigma(\hat{k})+\hat{\Omega}}{g_0})\right]/g_0^2 m$

$\check{Q} \equiv \left[m\sigma'\left(\frac{g_0\sigma(\check{k})+g_0\check{\Omega}}{m}\right)\sigma'(\check{k}) + m\zeta'(\check{k}) + \zeta'(\frac{g_0\sigma(\check{k})+g_0\check{\Omega}}{m})\right]/m^2.$

Long-run dynamics when $n = 0$: a special case

Remark

Assume $N_{-1} = N_0 > 0$ *and* $n = 0$ *and* $m = 1$.

- *Provided* $\sigma(0) = 0$, $\sigma'(k_t) > 0$, *as well as* $\zeta(0) = 0$, $\zeta'(k_t) > 0$, *we have that, if* $lim_{k\to 0}\, 1 - \sigma'(k_t) < lim_{k\to 0}\, \zeta'(k_t)$ *and* $lim_{k\to +\infty}\, 1 - \sigma'(k_t) > lim_{k\to +\infty}\, \zeta'(k_t)$, *there exists a stable stationary equilibrium.*

Fact

But in general, the timing of births affects the nature (stationary or cyclical) of the long-run dynamics of the economy.
=> focusing on the TFR can be misleading.

The long-run social optimum

- Assume that there exists a unique SSE (thus $n > 0$).
- The social planner selects the best feasible SSE (Samuelson 1975):

$$\max_{c,d,b,k,n,m} u(c) + \beta u(d) + \beta^2 u(b)$$

$$\text{s.t. } F\left(k, 1 + \frac{1}{g}\right) - gk = c + \frac{d}{g} + \frac{b}{g^2}$$

$$\text{where } g = \frac{n + \sqrt[2]{n^2 + 4m}}{2}.$$

- FOCs are:

$$\frac{u'(c^*)}{\beta u'(d^*)} = \frac{u'(d^*)}{\beta u'(b^*)} = g^* = \frac{n^* + \sqrt[2]{n^{*2} + 4m^*}}{2}$$

$$F_k(k^*, \cdot) = g^* = \frac{n^* + \sqrt[2]{n^{*2} + 4m^*}}{2}$$

- Optimal consumption path and capital (GR) depends on g^*.
- No one-to-one substituability between n and m.
  $=>$ the TFR $n + m$ is irrelevant.

The long-run social optimum

- FOCs for optimal n and m:

$$\begin{aligned} g_n^*\left[\frac{-F_L\left(k^*,\cdot\right)}{g^{*2}}-k^*+\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}\right] &= 0 \\ g_m^*\left[\frac{-F_L\left(k^*,\cdot\right)}{g^{*2}}-k^*+\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}\right] &= 0 \end{aligned}$$

where $g_{n^*}^* = \frac{1+n^*\left(n^{*2}+4m^*\right)^{-1/2}}{2} = \frac{1}{2}+\frac{n^*}{2}\frac{1}{\sqrt[2]{n^{*2}+4m^*}}$ and $g_{m^*}^* = \left(n^{*2}+4m^*\right)^{-1/2} = \frac{1}{\sqrt[2]{n^{*2}+4m^*}}$.

- Assuming an interior social optimum, so that the two FOCs are satisfied, it must be the case that:

$$k^*+\frac{F_L\left(k^*,\cdot\right)}{g^{*2}} = \frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}$$

- Optimal cohort growth such that capital dilution (Solow effect) (LHS) equals, at the margin, the intergenerational redistribution effect (Samuelson effect) (RHS).

The long-run social optimum

- There is no one-to-one substituability between early births and late births (except when replacement fertility is optimal).

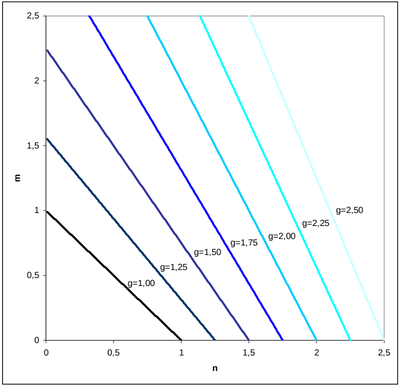


Figure: Substituability between n and m.

- If g^* is high, a low n requires a much higher m (and TFR).

The Serendipity Theorem (Samuelson 1975)

- In our framework, the agent's problem is, at the SE:

$$\max_{c,d,b} u(c) + \beta u(d) + \beta^2 u(b)$$

$$\text{s.t. } w + \frac{w}{R} = c + \frac{d}{R} + \frac{b}{R^2}$$

where $w = \left(F(k, \frac{1+g}{g}) - F_k(k, \frac{1+g}{g})k\right) \frac{g}{1+g}$ and $R = F_k(k, \frac{1+g}{g})$.

- The FOCs are:

$$\frac{u'(c)}{\beta u'(d)} = \frac{u'(d)}{\beta u'(b)} = R = F_k(k, \frac{1+g}{g})$$

- Imposing $g^* = \frac{n^* + \sqrt[2]{n^{*2} + 4m^*}}{2} = F_k(k^*, \frac{1+g^*}{g^*})$ generates the same FOCs as in the social planner's problem.

Fact

Assuming a unique SSE, imposing g^* through a pair (n, m) makes the competitive economy converge towards the long-run social optimum.

Extension 1: rational expectations about factor prices (1)

- Assuming $u(c) = log(c)$ and $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$, the dynamic system becomes, under $m > 0$:

$$
\begin{aligned}
k_{t+1} &\equiv \tilde{G}(k_t, X_t, g_t) = \frac{(\beta+\beta^2)Ak_t^{\alpha}\alpha(1-\alpha)\left(\frac{m}{g_t-n+m}\right)^{\alpha}(1+g_t)}{g_t\left[\left(1+\beta+\beta^2\right)\alpha\left(1+g_t\right)+\left(1-\alpha\right)\right]} \\
&+\frac{\beta^2A^2\alpha^2k_t^{\alpha-1}\left(\frac{m}{g_t-n+m}\right)^{\alpha-1}X_t\left(\frac{m(g_t-n)}{m-ng_t+n^2+m(g_t-n)}\right)^{\alpha}(1+g_t)}{\frac{g_tm}{g_t-n}\left[\left(1+\beta+\beta^2\right)\alpha\left(1+g_t\right)+\left(1-\alpha\right)\right]} \\
&+\frac{\beta^2A\alpha k_t^{\alpha}(1-\alpha)\left(\frac{m}{g_t-n+m}\right)^{\alpha}(1+g_t)}{\frac{g_tm}{g_t-n}\left[\left(1+\beta+\beta^2\right)\alpha\left(1+g_t\right)+\left(1-\alpha\right)\right]} \\
X_{t+1} &\equiv \tilde{H}(k_t) = (1-\alpha)k_t^{\alpha} \\
g_{t+1} &\equiv I(g_t) = n + \frac{m}{g_t}
\end{aligned}
$$

Extension 1: rational expectations about factor prices (2)

Proposition

Assume $N_{-1} > 0$ and $N_0 > 0$, as well as $m > 0$.

- *Provided* $\lim_{k\to\infty} \frac{(2-\alpha)k_t^{1-\alpha}g^2\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}{\beta^2 A^2\alpha^2\left(\frac{1+g}{g}\right)^{\alpha-1}(1+g)} > \frac{(1-\alpha)g[g+g\beta-1]}{A\alpha(1+g)}$ *with* $g = \frac{n+\sqrt[2]{n^2+4m}}{2}$, *there exists a stationary equilibrium.*
- *That stationary equilibrium is locally stable if and only if:*

(i) $\left|\frac{\Lambda\alpha m}{g^2}\right| < 1$

(ii) $1 > \frac{-\alpha m}{g^3}\left[1 - A\Lambda\right] + \Lambda\left(\frac{Am(1-\alpha)}{g^2} - \alpha\right) - \left[\alpha - \Lambda A - \frac{m}{g^2}\right]\left[\frac{\Lambda\alpha m}{g^2}\right] + \left[\frac{\Lambda\alpha m}{g^2}\right]^2$

(iii) $\frac{m\alpha[1-\Lambda A]}{g^2} - \Lambda\left(\frac{Am(1-\alpha)-\alpha g^2}{g^2}\right) - 1 < \frac{\alpha g^2 - A\Lambda g^2 - m(1-\alpha\Lambda)}{g^2} < \frac{-m\alpha[1-\Lambda A]}{g^2} + \Lambda\left(\frac{Am(1-\alpha)-\alpha g^2}{g^2}\right) + 1$

where $\Lambda \equiv \frac{\beta^2 A^2(1-\alpha)\alpha^2 k^{2\alpha-2}\left(\frac{m}{g-n+m}\right)^{\alpha-1}\left(\frac{m(g-n)}{m-ng+n^2+m(g-n)}\right)^{\alpha}(1+g)}{\frac{gm}{g-n}\left[\left(1+\beta+\beta^2\right)\alpha(1+g)+(1-\alpha)\right]}$.

Extension 1: rational expectations about factor prices (3)

Corollary

Assume $n = 0$ and $m > 0$.

- *If*
  $$\frac{(1-\alpha)\sqrt[2]{m}\left[\sqrt[2]{m}+\sqrt[2]{m}\beta-1\right]}{A\alpha\left(1+\sqrt[2]{m}\right)} < lim_{k\to\infty}\frac{(2-\alpha)k_t^{1-\alpha}m\left[\left(1+\beta+\beta^2\right)\alpha\left(1+\sqrt[2]{m}\right)+(1-\alpha)\right]}{\beta^2A^2\alpha^2\left(\frac{1+\sqrt[2]{m}}{\sqrt[2]{m}}\right)^{\alpha-1}\left(1+\sqrt[2]{m}\right)},$$
  there exists a stationary equilibrium with positive capital.
- *That equilibrium is not stable.*

Extension 2: endogenous fertility (1)

- Here children are *consumption goods* (unlike dynastic altruism in Pestieau & Ponthiere 2012):

$$\max_{c_t, d_{t+1}, b_{t+2}, n_t, m_{t+1}} \begin{cases} u(c_t) + v(n_t) + \beta u(d_{t+1}) \\ +\beta v(m_{t+1}) + \beta^2 u(b_{t+2}) \end{cases}$$

$$\text{s.t. } w_t + \frac{w_{t+1}}{R_{t+1}} = c_t + \theta n_t + \frac{d_{t+1} + \vartheta m_{t+1}}{R_{t+1}} + \frac{b_{t+2}}{R_{t+1} R_{t+2}}$$

where θ and ϑ are costs of resp. early and late children, while $v(\cdot)$ is increasing and concave. FOCs yield:

$$\frac{u'(c_t)}{u'(d_{t+1})} = \beta R_{t+1} \quad \text{and} \quad \frac{u'(d_{t+1})}{u'(b_{t+2})} = \beta R_{t+2}$$

as well as, for children:

$$\frac{v'(n_t)}{v'(m_{t+1})} = \frac{u'(c_t)\theta}{u'(d_{t+1})\vartheta} = \beta R_{t+1} \frac{\theta}{\vartheta}$$

Extension 2: endogenous fertility (2)

- Savings and fertility functions:

$$\begin{aligned} s_t &= S\left(R_{t+1}, R_{t+2}, w_t, w_{t+1}\right) \text{ and } z_{t+1} = Z((R_{t+1}, R_{t+2}, w_t, w_{t+1}) \\ n_t &= N(w_t, w_{t+1}, R_{t+1}, R_{t+2}) \text{ and } m_{t+1} = M(w_t, w_{t+1}, R_{t+1}, R_{t+2}) \end{aligned}$$

- Assuming myopic anticipations, the dynamics system becomes:

$$\begin{aligned} k_{t+1} &\equiv \hat{G}(k_t, \Omega_t, g_t) = \frac{\sigma\left(k_t\right)}{g_t} + \frac{\Omega_t}{g_t} \\ \Omega_{t+1} &\equiv \hat{H}\left(k_t\right) = \frac{\zeta(k_t)}{g_t} \\ g_{t+1} &\equiv \hat{I}(k_t, \Omega_t, g_t) = \eta(G\left(k_t, \Omega_t, g_t\right)) + \frac{\mu(k_t)}{g_t} \end{aligned}$$

where $s_t = \sigma(k_t)$, $z_{t+1} = \zeta(k_t)$, $n_t = \eta(k_t)$ and $m_{t+1} = \mu(k_t)$.

Extension 2: endogenous fertility (3)

Corollary

Assume that there exists a stationary equilibrium with $\eta(k^*) = 0$.

- *That equilibrium is locally stable if and only if:*

(i) $\left|\frac{\zeta'(k^*)\mu(k^*)-\zeta(k^*)\mu'(k^*)}{g^{*4}}\right| < 1$;

(ii) $1 >$

$$\frac{\zeta(k^*)\eta'(k^*)}{g^{*3}} + \frac{-\sigma'(k^*)\mu(k^*)}{g^{*3}} + \frac{\mu'(k^*)[\sigma(k^*)+\Omega^*]}{g^{*3}} - \frac{\zeta'(k^*)}{g^{*2}}$$
$$-\left[\frac{\zeta'(k^*)\mu(k^*)-\zeta(k^*)\mu'(k^*)}{g^{*4}}\right]\left[\frac{\sigma'(k^*)}{g^*} - \eta'(k^*)\left[\frac{\sigma(k^*)+\Omega^*}{g^{*2}}\right] - \frac{\mu(k^*)}{g^{*2}}\right] + \left[\frac{\zeta'(k^*)\mu(k^*)-\zeta(k^*)\mu'(k^*)}{g^{*4}}\right]^2$$

(iii) $\frac{\sigma'(k^*)}{g^*} - \frac{\zeta(k^*)\eta'(k^*)}{g^{*3}} - \frac{\mu'(k^*)\sigma(k^*)}{g^{*3}} - \frac{\mu'(k^*)\zeta(k^*)}{g^{*4}} + \frac{\zeta'(k^*)}{g^{*2}} - 1$

$$< \frac{\sigma'(k^*)}{g^*} - \frac{\zeta(k^*)\eta'(k^*)}{g^{*3}} - \frac{\eta'(k^*)\sigma(k^*)}{g^{*2}} - \frac{\mu'(k^*)\zeta(k^*)}{g^{*4}} + \frac{\zeta'(k^*)}{g^{*2}} - 1$$
$$< -\frac{\sigma'(k^*)}{g^*} + \frac{\zeta(k^*)\eta'(k^*)}{g^{*3}} + \frac{\mu'(k^*)\sigma(k^*)}{g^{*3}} + \frac{\mu'(k^*)\zeta(k^*)}{g^{*4}} - \frac{\zeta'(k^*)}{g^{*2}} + 1.$$

- Stability is possible, but birth timing still matters (through $\eta'(\cdot), \mu'(\cdot)$).

Extension 2: endogenous fertility (4)

- The new social planner's problem:

$$\max_{c,d,b,k,n,m} u(c)+v(n)+\beta u(d)+\beta v(m)+\beta^2 u(b)$$

$$\text{s.t. } F\left(k,1+\frac{1}{g}\right)-kg=c+\theta n+\frac{d}{g}+\frac{\vartheta m}{g}+\frac{b}{g^2}$$

where $g=\frac{n+\sqrt[2]{n^2+4m}}{2}$. FOCs yield:

$$F_k\left(k^*,\cdot\right)=\frac{n^*+\sqrt[2]{n^{*2}+4m^*}}{2}=g^*=\frac{u'(c^*)}{\beta u'(d^*)}=\frac{u'(d^*)}{\beta u'(b^*)}$$

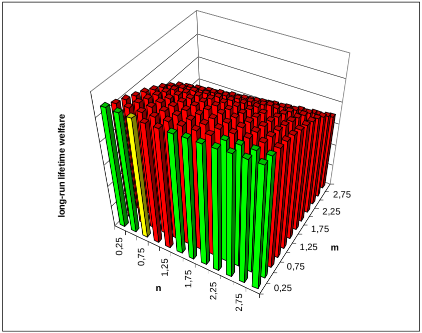
$$g_n\left(F_L\left(\cdot\right)\frac{1}{g^{*2}}+k^*\right)+\theta=\frac{v'(n^*)}{u'(c^*)}+g_n\left[\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}+\frac{\vartheta m^*}{g^{*2}}\right]$$

$$g_m\left(F_L\left(\cdot\right)\frac{1}{g^{*2}}+k^*\right)+\frac{\vartheta}{g^*}=\frac{\beta v'(m^*)}{u'(c^*)}+g_m\left[\frac{d^*}{g^{*2}}+\frac{2b^*}{g^{*3}}+\frac{\vartheta m^*}{g^{*2}}\right]$$

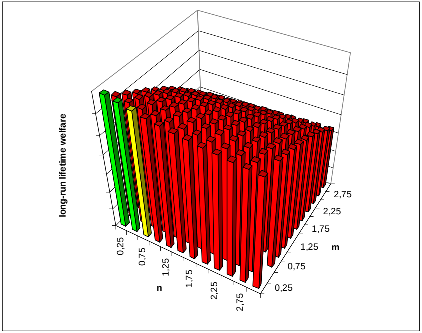
- Here birth timing matters beyond getting optimal g^*.
- The Serendipity Theorem is no longer valid here.

Extension 2: endogenous fertility (5)

- Two calibrations rationalizing $n = 0.8$, $m = 0.2$ (in yellow) assuming $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$ with $A = 10$, $\alpha = 0.3$, and $u(c) = log(c)$, $v(n) = \varphi log(n)$ with $\varphi = 0.05$.
- Green (red) = higher (lower) SS lifetime welfare than under current (n, m).



$\beta = 0.80$, $\theta = 0.18$, $\vartheta = 2.10$.



$\beta = 0.60$, $\theta = 0.22$, $\vartheta = 1.95$.

- Ambiguous gains from raising fertility. Delaying births is not optimal.

Extension 3: three reproduction periods (1)

- Consider now a 5-period OLG with 3 reproduction periods.
- The number of agents born at time i is now:

$$N_t = N_{t-1} n + N_{t-2} m + N_{t-3} o$$

Dividing this by N_{t-1}, we obtain:

$$g_t = n + \frac{m}{g_{t-1}} + \frac{o}{g_{t-1} g_{t-2}}$$

- The dynamics of the population is given by the following two-dimensional dynamic system:

$$\begin{aligned} g_{t+1} &= n + \frac{m}{g_t} + \frac{\ell_t}{g_t} \\ \ell_{t+1} &= \frac{o}{g_t} \end{aligned}$$

Extension 3: three reproduction periods (2)

- The long-run cohort growth factor g satisfies:

$$g^3 - ng^2 - mg - o = 0$$

- Convergence towards equilibrium g still depends on fertility timing.

Fact

Assume $n = 0$ and $TFR \neq 1$.

	$o = 0$	$o > 0$
$m = 0$	$TFR = 0$	*no convergence*
$m > 0$	*no convergence*	*convergence*

- Hence the Postulate 2 in MacFarland's (1969) discrete time model (existence of 2 strictly positive age-specific fertility rates) is *necessary* to have the asymptotic convergence of the age-structure.

Extension 3: three reproduction periods (3)

- But asymptotic convergence does not imply birth timing is neutral! (two fertility profiles with TFR $= 1.05$, but transition: 16 p. $<<$ 5,750 p.)

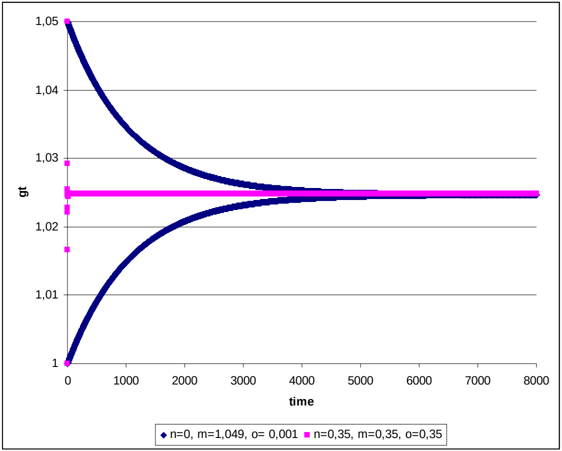


Figure: Asymptotic convergence of g_t.

Conclusions

- For an equal TFR $n + m$, the timing of births matters for long-run economic dynamics.
- From the perspective of long-run social welfare, there is no one-to-one substituability between early and late births.
- Robustness of (most) results to various aspects of the modelling (expectations, choices, number of periods).